

PRECOSHIEVES OF PRO-SETS AND ABELIAN PRO-GROUPS ARE SMOOTH

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ABSTRACT. Let \mathbb{D} be the category of pro-sets (or abelian pro-groups). It is proved that for any Grothendieck site X , there exists a reflector from the category of precosheaves on X with values in \mathbb{D} to the full subcategory of cosheaves. In the case of precosheaves on topological spaces, it is proved that any precosheaf is smooth, i.e. is locally isomorphic to a cosheaf. Locally constant cosheaves are constructed, and there are established connections with shape theory.

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0. INTRODUCTION

While the theory of *sheaves* is well developed, and is covered in a plenty of publications, the theory of *cosheaves* is represented much poorer. The main reason for this is that *cofiltrant limits* are *not* exact in the “usual” categories like sets or abelian groups. On the contrary, *filtrant colimits* are exact, which allows to construct rather rich theories of sheaves (of sets or of abelian groups). Since cosheaves

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of sets (respectively, of abelian groups) are in fact sheaves with values in the opposite category \mathbf{SET}^{op} (respectively, \mathbf{AB}^{op}), the above argument would mean that \mathbf{SET}^{op} and \mathbf{AB}^{op} are badly suited for sheaf theory.

The first step in building a suitable theory of cosheaves would be constructing a *cosheaf associated with a precosheaf*. It is impossible in general, because of the mentioned drawbacks of categories \mathbf{SET}^{op} and \mathbf{AB}^{op} . In [Bre97] and [Bre68], it is made an attempt to avoid this difficulty by introducing the so-called *smooth* precosheaves (Definition 1.5). It is not clear whether one has enough smooth precosheaves for building a suitable theory of cosheaves. Another difficulty is the lack of suitable *locally constant* cosheaves. In [Bre97] and [Bre68], such cosheaves are constructed only for locally connected spaces.

There is a lot of papers (that are not cited here) dealing with sheaves with values in a general category \mathbb{D} having suitable properties. Such a sheaf theory would allow constructing a suitable cosheaf theory with values in \mathbb{D}^{op} . An interesting attempt is made in [Sch87] where the author sketches a sheaf theory on topological spaces with values in the category of abelian ind-groups $Ind(\mathbf{AB}^{op})$, which is equivalent to a cosheaf theory with values in the category of abelian pro-groups

$$Pro(\mathbf{AB}) \approx (Ind(\mathbf{AB}^{op}))^{op}.$$

The latter category, as well as the category of pro-sets $Pro(\mathbf{SET})$, seems to be one of the best candidates for a suitable cosheaf theory.

In this paper, we begin a systematic study of cosheaves on topological spaces (as well as on general Grothendieck sites) with values in $Pro(\mathbf{SET})$ and $Pro(\mathbf{AB})$ (see Appendix, Section 4). In Theorems 1.1 and 1.2, the cosheaf $\mathcal{A}_\#$ associated with a precosheaf \mathcal{A} is constructed, giving a pair of adjoint functors and a reflector from the category of precosheaves to the category of cosheaves. It appeared that on a topological space such a cosheaf is *locally isomorphic* to the original precosheaf (Theorems 1.3 and 1.4), implying that any precosheaf is smooth (Corollary 1.6). In Theorem 1.7, locally constant cosheaves are constructed. It turns out that they are closely connected to shape theory. Namely, the locally constant cosheaf $(S^{LC})_\#$ with values in $Pro(\mathbf{SET})$ is isomorphic to the pro-homotopy cosheaf $S \times pro-\pi_0$, while the locally constant cosheaf $(A^{LC})_\#$ with values in $Pro(\mathbf{AB})$ is isomorphic to the pro-homology cosheaf $pro-H_0(-, A)$.

In future papers, we are planning to develop homology of cosheaves, i.e. to study projective and flabby cosheaves, projective and flabby resolutions, and to construct the left satellites

$$H_n(X, \mathcal{A}) := L_n \Gamma(X, \mathcal{A})$$

of the global sections functor

$$H_0(X, \mathcal{A}) := \Gamma(X, \mathcal{A}).$$

It is expected that deeper connections to shape theory will be discovered, as is stated in the two Conjectures below:

Conjecture 0.1. *The left satellites of H_0 are naturally isomorphic to the pro-homology:*

$$H_n(X, pro-H_0(-, A)) \approx pro-H_n(X, A).$$

Conjecture 0.2. *The **non-abelian** left satellites of H_0 are naturally isomorphic to the pro-homotopy:*

$$H_n(X, S \times \text{pro-}\pi_0) \approx S \times \text{pro-}\pi_n(X).$$

It is not yet clear how to generalize the above Conjectures to *strong shape theory*. However, we have some ideas how to do that.

1. MAIN RESULTS

Let X be a site (Definition 2.1), and let $\text{PCS}(X, \text{Pro}(\text{SET}))$ and $\text{CS}(X, \text{Pro}(\text{SET}))$ be the categories of precosheaves and cosheaves, respectively, on X , with values in $\text{Pro}(\text{SET})$ (Definition 2.8). Let further $\text{PCS}(X, \text{Pro}(\mathbb{A}\mathbb{B}))$ and $\text{CS}(X, \text{Pro}(\mathbb{A}\mathbb{B}))$ be the categories of precosheaves and cosheaves, respectively, on X , with values in $\text{Pro}(\mathbb{A}\mathbb{B})$ (Definition 2.8). See Appendix (Section 4) for the definition and properties of the categories $\text{Pro}(\text{SET})$ and $\text{Pro}(\mathbb{A}\mathbb{B})$.

Theorem 1.1. (1) *The inclusion functor*

$$I : \text{CS}(X, \text{Pro}(\text{SET})) \longrightarrow \text{PCS}(X, \text{Pro}(\text{SET}))$$

has a right adjoint

$$()_{\#} : \text{PCS}(X, \text{Pro}(\text{SET})) \longrightarrow \text{CS}(X, \text{Pro}(\text{SET})).$$

(2) *For any cosheaf \mathcal{A} on X with values in $\text{Pro}(\text{SET})$, the canonical morphism*

$$\mathcal{A}_{\#} \longrightarrow \mathcal{A}$$

is an isomorphism of cosheaves, i.e. $()_{\#}$ is a reflector from the category of precosheaves with values in $\text{Pro}(\text{SET})$ to the full subcategory of cosheaves with values in the same category.

Theorem 1.2. (1) *The inclusion functor*

$$I : \text{CS}(X, \text{Pro}(\mathbb{A}\mathbb{B})) \longrightarrow \text{PCS}(X, \text{Pro}(\mathbb{A}\mathbb{B}))$$

has a right adjoint

$$()_{\#} : \text{PCS}(X, \text{Pro}(\mathbb{A}\mathbb{B})) \longrightarrow \text{CS}(X, \text{Pro}(\mathbb{A}\mathbb{B})).$$

(2) *For any cosheaf \mathcal{A} on X with values in $\text{Pro}(\mathbb{A}\mathbb{B})$, the canonical morphism*

$$\mathcal{A}_{\#} \longrightarrow \mathcal{A}$$

is an isomorphism of cosheaves, i.e. $()_{\#}$ is a reflector from the category of precosheaves with values in $\text{Pro}(\mathbb{A}\mathbb{B})$ to the full subcategory of cosheaves with values in the same category.

In Theorems 1.3, 1.4, 1.7, and in Corollary 1.6 below X is a topological space. We denote by the same letter X the corresponding site (Example 2.2).

Theorem 1.3. (1) *For any precosheaf \mathcal{A} on X with values in $\text{Pro}(\text{SET})$,*

$$\mathcal{A}_{\#} \longrightarrow \mathcal{A}$$

is a local isomorphism (Definition 2.21).

(2) *Any local isomorphism*

$$\mathcal{A} \longrightarrow \mathcal{B}$$

between cosheaves on X with values in $\text{Pro}(\text{SET})$, is an isomorphism.

(3) *If $\mathcal{B} \longrightarrow \mathcal{A}$ is a local isomorphism, and \mathcal{B} is a cosheaf, then $\mathcal{B} \approx \mathcal{A}_{\#}$.*

Theorem 1.4. (1) For any precosheaf \mathcal{A} on X with values in $\text{Pro}(\mathbb{A}\mathbb{B})$,

$$\mathcal{A}_{\#} \longrightarrow \mathcal{A}$$

is a local isomorphism.

(2) Any local isomorphism

$$\mathcal{A} \longrightarrow \mathcal{B}$$

between cosheaves on X with values in $\text{Pro}(\mathbb{A}\mathbb{B})$, is an isomorphism.

(3) If $\mathcal{B} \longrightarrow \mathcal{A}$ is a local isomorphism, and \mathcal{B} is a cosheaf, then $\mathcal{B} \approx \mathcal{A}_{\#}$.

The two latter Theorems guarantee that all precosheaves are smooth:

Definition 1.5. ([Bre97], Corollary VI.3.2 and Definition VI.3.4, or [Bre68], Corollary 3.5 and Definition 3.7) A precosheaf \mathcal{A} is called **smooth** iff there exist precosheaves \mathcal{B} and \mathcal{B}' , a cosheaf \mathcal{C} , and local isomorphisms

$$\mathcal{A} \longrightarrow \mathcal{B} \longleftarrow \mathcal{C},$$

or, equivalently, local isomorphisms

$$\mathcal{A} \longleftarrow \mathcal{B}' \longrightarrow \mathcal{C}.$$

Corollary 1.6. Any precosheaf with values in $\text{Pro}(\mathbb{S}\mathbb{E}\mathbb{T})$ or in $\text{Pro}(\mathbb{A}\mathbb{B})$ is smooth.

Proof.

$$\mathcal{A} \xrightarrow{\text{Id}} \mathcal{A} \longleftarrow \mathcal{A}_{\#}$$

or

$$\mathcal{A} \longleftarrow \mathcal{A}_{\#} \xrightarrow{\text{Id}} \mathcal{A}_{\#}.$$

□

We are now able to construct *locally constant* cosheaves, and to establish connections to shape theory.

Theorem 1.7. Let S be a set, and let A be an abelian group.

(1) The precosheaf

$$\mathcal{P}(U) := S \times \text{pro-}\pi_0(U)$$

where $\text{pro-}\pi_0$ is the pro-homotopy functor from [MS82], p. 130, is a cosheaf.

(2) Let S^{LC} be the locally constant precosheaf corresponding to S (Definition 2.16) on X with values in $\text{Pro}(\mathbb{S}\mathbb{E}\mathbb{T})$. Then $(S^{LC})_{\#}$ is naturally isomorphic to \mathcal{P} .

(3) The precosheaf

$$\mathcal{H}(U) := \text{pro-}H_0(U, A)$$

where $\text{pro-}H_0$ is the pro-homology functor from [MS82], p. 121, is a cosheaf.

(4) Let A^{LC} be the locally constant precosheaf corresponding to A (Definition 2.17) on X with values in $\text{Pro}(\mathbb{A}\mathbb{B})$. Then $(A^{LC})_{\#}$ is naturally isomorphic to \mathcal{H} .

2. COSHEAVES AND PRECOSHEAVES

2.1. Cosheaves and precosheaves with values in $Pro(\mathbf{SET})$.

Definition 2.1. A **Grothendieck site** (or simply a **site**) (see [SGA72], [KS06], Definition 16.1.5, or [Tam94], p. 24) is a pair

$$X = (Cat(X), Cov(X))$$

where $Cat(X)$ is a small category (Definition 4.1), and $Cov(X)$ is a collection of families of morphisms

$$\{U_i \longrightarrow U\} \in Cat(X)$$

satisfying COV1-COV4 from [KS06], p. 391, or T1-T3 from [Tam94], Definition I.1.2.1.

Example 2.2. Let X be a topological space. Let us consider a site denoted by the same letter

$$X = (Cat(X), Cov(X)).$$

$Cat(X)$ will consist of open subsets of X as objects and inclusions $U \subseteq V$ as morphisms. The set of coverings $Cov(X)$ consists of families

$$\{U_i \longrightarrow U\} \in Cat(X)$$

with

$$\bigcup_i U_i = U.$$

Let \mathbb{D} be a category.

Definition 2.3. Let $X = (Cat(X), Cov(X))$ be a site. A **precosheaf** on X with values in \mathbb{D} is a covariant functor \mathcal{A} from $Cat(X)$ to \mathbb{D} . A **presheaf** on X with values in \mathbb{D} is a contravariant functor \mathcal{A} from $Cat(X)$ to \mathbb{D} . **Morphisms** between precosheaves (presheaves) are morphisms between the corresponding functors.

Assume \mathbb{D} admits small coproducts.

Definition 2.4. A precosheaf \mathcal{A} on a site $X = (Cat(X), Cov(X))$ with values in \mathbb{D} is called **coseparated** (**epiprecoshief** in the terminology of [Bre68] and [Bre97]) iff

$$\coprod_i \mathcal{A}(U_i) \longrightarrow \mathcal{A}(U)$$

is an epimorphism for any covering $\{U_i \longrightarrow U\} \in Cov(X)$.

Assume \mathbb{D} is **cocomplete** (Definition 4.5).

Definition 2.5. A **cosheaf** on a site $X = (Cat(X), Cov(X))$ with values in \mathbb{D} is a precosheaf \mathcal{A} such that

$$coker \left(\coprod_{i,j} \mathcal{A}(U_i \times_U U_j) \rightrightarrows \coprod_i \mathcal{A}(U_i) \right) \approx \mathcal{A}(U)$$

for any covering $\{U_i \longrightarrow U\} \in Cov(X)$.

Assume \mathbb{D} admits small products.

Definition 2.6. A presheaf \mathcal{A} on a site $X = (Cat(X), Cov(X))$ with values in \mathbb{D} is called **separated** (**monopreshief** in the terminology of [Bre97]) iff

$$\mathcal{A}(U) \longrightarrow \prod_i \mathcal{A}(U_i)$$

is a monomorphism for any covering $\{U_i \longrightarrow U\} \in Cov(X)$.

Assume \mathbb{D} is **complete** (Definition 4.5).

Definition 2.7. A **sheaf** on a site $X = (Cat(X), Cov(X))$ with values in \mathbb{D} is a presheaf \mathcal{A} such that

$$\mathcal{A}(U) \approx \ker \left(\prod_i \mathcal{A}(U_i) \rightrightarrows \prod_{i,j} \mathcal{A}(U_i \times_U U_j) \right)$$

for any covering $\{U_i \longrightarrow U\} \in Cov(X)$.

Let

$$X = (Cat(X), Cov(X))$$

be a site. We introduce the main categories of pre(co)sheaves and (co)sheaves.

Definition 2.8. Let us denote:

- a) by $\mathbf{PCS}(X, \mathbb{D})$ the category of precosheaves on X with values in \mathbb{D} ;
- b) by $\mathbf{PS}(X, \mathbb{D})$ the category of presheaves on X with values in \mathbb{D} ;
- c) (\mathbb{D} is cocomplete) by $\mathbf{CS}(X, \mathbb{D})$ the full subcategory of $\mathbf{PCS}(X, \mathbb{D})$ consisting of cosheaves;
- d) (\mathbb{D} is complete) by $\mathbf{S}(X, \mathbb{D})$ the full subcategory of $\mathbf{PS}(X, \mathbb{D})$ consisting of sheaves.

Definition 2.9. Given a precosheaf

$$\mathcal{A} : Cat(X) \longrightarrow Pro(\mathbf{SET})$$

on X , define a presheaf

$$\mathcal{A}^{op} : (Cat(X))^{op} \longrightarrow (Pro(\mathbf{SET}))^{op} \subseteq \mathbf{SET}^{\mathbf{SET}}$$

by

$$\begin{aligned} \mathcal{A}^{op}(U) &:= \mathcal{A}(U), U \in Cat(X), \\ \mathcal{A}^{op}(f^{op} : V \longrightarrow U) &:= (\mathcal{A}(f) : \mathcal{A}(V) \longrightarrow \mathcal{A}(U))^{op}. \end{aligned}$$

Definition 2.10. Given a presheaf \mathcal{B} on X with values in $(Pro(\mathbf{SET}))^{op}$, let $\iota(\mathcal{B})$ be the following presheaf on X with values in $\mathbf{SET}^{\mathbf{SET}}$:

$$\iota(\mathcal{B})(U) := \iota(\mathcal{B}(U))$$

where ι is the Ioneda embedding from Definition 4.9.

Given a precosheaf \mathcal{A} on X with values in $Pro(\mathbf{SET})$, let $\kappa(\mathcal{A})$ be the following presheaf on X with values in $\mathbf{SET}^{\mathbf{SET}}$:

$$\kappa(\mathcal{A})(U) := \kappa(\mathcal{B}(U))$$

where κ is the contravariant embedding from Definition 4.10.

The two Propositions below establish connections between coseparated precosheaves and separated presheaves, and connections between cosheaves and sheaves.

Proposition 2.11. The following conditions are equivalent:

- (1) \mathcal{A} is coseparated;
- (2) \mathcal{A}^{op} is a separated presheaf with values in $(Pro(\mathbb{SET}))^{op}$;
- (3) $\iota(\mathcal{A}^{op}) = \kappa(\mathcal{A})$ is a separated presheaf with values in $\mathbb{SET}^{\mathbb{SET}}$;
- (4) $\iota(\mathcal{A}^{op})(Z) = \kappa(\mathcal{A})(Z)$ is a separated presheaf of sets for any $Z \in \mathbb{SET}$.

Proof. Let

$$\{U_i \longrightarrow U\} \in Cov(X)$$

be a covering.

1 \iff 2: By duality,

$$\prod_i \mathcal{A}(U_i) \xrightarrow{\varphi} \mathcal{A}(U)$$

is an epimorphism of pro-sets iff

$$\mathcal{A}^{op}(U) \xrightarrow{\varphi^{op}} \prod_i \mathcal{A}^{op}(U_i)$$

is a monomorphism in $(Pro(\mathbb{SET}))^{op}$.

1 \implies 3. It is given that

$$\prod_i \mathcal{A}(U_i) \xrightarrow{\varphi} \mathcal{A}(U)$$

is an epimorphism of pro-sets. Then, for any set Z , the mapping

$$\kappa(\mathcal{A}(U))(Z) \xrightarrow{\kappa(\varphi)} \kappa\left(\prod_i \mathcal{A}(U_i)\right)(Z) \approx \prod_i \kappa(\mathcal{A}(U_i))(Z)$$

is a monomorphism (see Proposition 4.12). Therefore, $\kappa(\mathcal{A})$ is separated as a presheaf in $\mathbb{SET}^{\mathbb{SET}}$.

3 \implies 1. It is given that

$$\kappa(\mathcal{A}(U))(Z) \xrightarrow{\kappa(\varphi)} \prod_i \kappa(\mathcal{A}(U_i))(Z)$$

is a monomorphism for any set Z . Let (Z_s) be a pro-set. Then

$$\begin{aligned} Hom_{Pro(\mathbb{SET})}(\mathcal{A}(U), (Z_s)) &\approx \lim_s \kappa(\mathcal{A}(U))(Z_s) \xrightarrow{\lim_s \varphi(Z_s)} \\ \lim_s \prod_i \kappa(\mathcal{A}(U_i), Z_s) &= Hom_{Pro(\mathbb{SET})}\left(\prod_i \mathcal{A}(U_i), (Z_s)\right) \end{aligned}$$

is a monomorphism since limits in \mathbb{SET} convert monomorphisms to monomorphisms. It follows that

$$\prod_i \mathcal{A}(U_i) \xrightarrow{\varphi} \mathcal{A}(U)$$

is an epimorphism of pro-sets.

4 \iff 3. Follows from the fact that

$$\mathcal{B} \longrightarrow \mathcal{C}$$

is a monomorphism in $\mathbb{SET}^{\mathbb{SET}}$ iff

$$\mathcal{B}(Z) \longrightarrow \mathcal{C}(Z)$$

is a monomorphism for any $Z \in \mathbb{SET}$. □

Proposition 2.12. *The following conditions are equivalent:*

- (1) \mathcal{A} is a cosheaf;
- (2) \mathcal{A}^{op} is a sheaf with values in $(Pro(\mathbf{SET}))^{op}$;
- (3) $\iota(\mathcal{A}^{op}) = \kappa(\mathcal{A})$ is a sheaf with values in $\mathbf{SET}^{\mathbf{SET}}$.
- (4) $\iota(\mathcal{A}^{op})(Z) = \kappa(\mathcal{A})(Z)$ is a sheaf of sets for any $Z \in \mathbf{SET}$.

Proof. Let

$$\{U_i \longrightarrow U\} \in Cov(X)$$

be a covering.

1 \iff 2: By duality,

$$\left(\prod_{i,j} \mathcal{A}(U_i \times_U U_j) \rightrightarrows \prod_i \mathcal{A}(U_i) \right) \longrightarrow \mathcal{A}(U)$$

is a cokernel in $Pro(\mathbf{SET})$ iff

$$\mathcal{A}^{op}(U) \longrightarrow \left(\prod_i \mathcal{A}^{op}(U_i) \rightrightarrows \prod_{i,j} \mathcal{A}^{op}(U_i \times_U U_j) \right)$$

is a kernel in $(Pro(\mathbf{SET}))^{op}$.

1 \implies 3. It is given that

$$\left(\prod_{i,j} \mathcal{A}(U_i \times_U U_j) \rightrightarrows \prod_i \mathcal{A}(U_i) \right) \longrightarrow \mathcal{A}(U)$$

is a cokernel in $Pro(\mathbf{SET})$. Apply κ :

$$\kappa(\mathcal{A}(U)) \longrightarrow \left(\prod_i \kappa(\mathcal{A}(U_i)) \rightrightarrows \prod_{i,j} \kappa(\mathcal{A}(U_i \times_U U_j)) \right)$$

is a kernel in $\mathbf{SET}^{\mathbf{SET}}$ (Proposition 4.12). Therefore, $\kappa(\mathcal{A})$ is a sheaf in $\mathbf{SET}^{\mathbf{SET}}$.

3 \implies 1. It is given that, for any set Z , the mapping

$$\kappa(\mathcal{A}(U))(Z) \longrightarrow \left(\prod_i \kappa(\mathcal{A}(U_i))(Z) \rightrightarrows \prod_{i,j} \kappa(\mathcal{A}(U_i \times_U U_j))(Z) \right)$$

is a kernel in \mathbf{SET} . Let (Z_s) be any pro-set. Then, since limits commute with kernels and products and due to Proposition 4.12,

$$\begin{aligned} Hom_{Pro(\mathbf{SET})}(\mathcal{A}(U), (Z_s)) &= \lim_s \kappa(\mathcal{A}(U))(Z_s) = \\ &= \lim_s \ker \left(\prod_i \kappa(\mathcal{A}(U_i))(Z_s) \rightrightarrows \prod_{i,j} \kappa(\mathcal{A}(U_i \times_U U_j))(Z_s) \right) = \\ &= \ker \left(\prod_i \lim_s \kappa(\mathcal{A}(U_i))(Z_s) \rightrightarrows \prod_{i,j} \lim_s \kappa(\mathcal{A}(U_i \times_U U_j))(Z_s) \right) = \\ &= \ker \left(\prod_i Hom_{Pro(\mathbf{SET})}(\mathcal{A}(U_i), (Z_s)) \rightrightarrows \prod_{i,j} Hom_{Pro(\mathbf{SET})}(\mathcal{A}(U_i \times_U U_j), (Z_s)) \right). \end{aligned}$$

It follows that

$$\left(\coprod_{i,j} \mathcal{A}(U_i \times_U U_j) \rightrightarrows \coprod_i \mathcal{A}(U_i) \right) \longrightarrow \mathcal{A}(U)$$

is a cokernel in $Pro(\mathbf{SET})$.

4 \iff 3. Follows from the fact that

$$B \longrightarrow (C \rightrightarrows D)$$

is a kernel in $\mathbf{SET}^{\mathbf{SET}}$ iff

$$B(Z) \longrightarrow (C(Z) \rightrightarrows D(Z))$$

is a kernel for any $Z \in \mathbf{SET}$. □

2.2. Cosheaves and precosheaves with values in $Pro(\mathbb{A}\mathbb{B})$. Let

$$X = (Cat(X), Cov(X))$$

be a site.

Definition 2.13. *Given a presheaf*

$$\mathcal{A} : Cat(X) \longrightarrow Pro(\mathbb{A}\mathbb{B})$$

on X , define a presheaf

$$\mathcal{A}^{op} : (Cat(X))^{op} \longrightarrow (Pro(\mathbb{A}\mathbb{B}))^{op} \subseteq \mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$$

by

$$\begin{aligned} \mathcal{A}^{op}(U) &: = \mathcal{A}(U), U \in Cat(X), \\ \mathcal{A}^{op}(f^{op} : V \longrightarrow U) &: = (\mathcal{A}(f) : \mathcal{A}(V) \longrightarrow \mathcal{A}(U))^{op}. \end{aligned}$$

Proposition 2.14. *The following conditions are equivalent:*

- (1) \mathcal{A} is coseparated;
- (2) \mathcal{A}^{op} is a separated presheaf in $(Pro(\mathbb{A}\mathbb{B}))^{op}$;
- (3) $\iota(\mathcal{A}^{op}) = \kappa(\mathcal{A})$ is a separated presheaf in $\mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$.
- (4) $\iota(\mathcal{A}^{op})(Z) = \kappa(\mathcal{A})(Z)$ is a separated presheaf in AB for any $Z \in AB$.

Proof. Just repeat the proof of Proposition 2.11. Remember that in that Proposition κ means the full contravariant embedding

$$\kappa : Pro(\mathbf{SET}) \longrightarrow \mathbf{SET}^{\mathbf{SET}}$$

from Definition 4.10, while in this Proposition κ means the full contravariant embedding

$$\kappa : Pro(\mathbb{A}\mathbb{B}) \longrightarrow \mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$$

from Definition 4.15. □

Proposition 2.15. *The following conditions are equivalent:*

- (1) \mathcal{A} is a cosheaf;
- (2) \mathcal{A}^{op} is a sheaf in $(Pro(\mathbb{A}\mathbb{B}))^{op}$;
- (3) $\iota(\mathcal{A}^{op}) = \kappa(\mathcal{A})$ is a sheaf in $\mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$.
- (4) $\iota(\mathcal{A}^{op})(Z) = \kappa(\mathcal{A})(Z)$ is a sheaf in AB for any $Z \in AB$.

Proof. Just repeat the proof of Proposition 2.12. □

2.3. Cosheaves and precosheaves on topological spaces. Throughout this Section, X is a topological space considered as a site (see Example 2.2).

Definition 2.16. Let S be a set. We denote by S^{LC} the following **locally constant precosheaf** on X :

$$S^{LC}(U) := \begin{cases} S & \text{if } U \neq \emptyset; \\ \emptyset & \text{if } U = \emptyset. \end{cases}$$

Definition 2.17. Let A be an abelian group. Analogously to Definition 2.16, denote by A^{LC} the following precosheaf on X :

$$A^{LC}(U) := \begin{cases} A & \text{if } U \neq \emptyset; \\ 0 & \text{if } U = \emptyset. \end{cases}$$

To introduce local isomorphisms, one needs the notion of a *costalk*, which is dual to the notion of a *stalk* in sheaf theory.

Definition 2.18. Let \mathbb{D} and \mathbb{E} be categories, and assume that \mathbb{D} admits cofiltrant limits, and that \mathbb{E} admits filtrant colimits. Let $x \in X$ be a point. Let further \mathcal{A} be a precosheaf on X with values in \mathbb{D} , and \mathcal{B} be a presheaf on X with values in \mathbb{E} . Denote

$$\begin{aligned} \mathcal{A}^x &:= \lim_{x \in U} \mathcal{A}(U), \\ \mathcal{B}_x &:= \operatorname{colim}_{x \in U} \mathcal{B}(U). \end{aligned}$$

We will call \mathcal{A}^x the **costalk** of \mathcal{A} at x , and \mathcal{B}_x the **stalk** of \mathcal{B} at x .

Example 2.19. If \mathcal{A} is a precosheaf of sets on X , then \mathcal{A}^x is the limit $\lim_{x \in U} \mathcal{A}(U)$ in \mathbf{SET} . However, if the same precosheaf is considered as a precosheaf of pro-sets, then \mathcal{A}^x is the pro-set defined by the cofiltrant system

$$\mathcal{A}^x = (\mathcal{A}(U) : x \in U).$$

Example 2.20. Let \mathcal{A} is a precosheaf of abelian groups on X . According to [Bre68], p. 5, or [Bre97], p. 420, \mathcal{A} is called **locally zero** iff for any $x \in X$ and any open neighborhood U of x there exists another open neighborhood V ,

$$x \in V \subseteq U$$

such that

$$\mathcal{A}(V) \longrightarrow \mathcal{A}(U)$$

is zero. If we consider, however, the precosheaf \mathcal{A} as a precosheaf of abelian pro-groups, then \mathcal{A} is locally zero iff for any $x \in X$, \mathcal{A}^x is the zero object in the category $\operatorname{Pro}(\mathbb{A}\mathbb{B})$.

Definition 2.21. Let

$$\mathcal{A} \longrightarrow \mathcal{B}$$

be a morphism of precosheaves on X . It is called a **local isomorphism** iff

$$\mathcal{A}^x \longrightarrow \mathcal{B}^x$$

is an isomorphism for any $x \in X$.

Remark 2.22. It is clear that a morphism $\mathcal{A} \longrightarrow \mathcal{B}$ of precosheaves of abelian groups is a local isomorphism in the sense of the definitions from [Bre68], p. 6, or [Bre97], p. 421, iff it is a local isomorphism in our sense when considered as a morphism of precosheaves with values in $\operatorname{Pro}(\mathbb{A}\mathbb{B})$.

Definition 2.23. *Let*

$$\mathcal{A} \longrightarrow \mathcal{B}$$

*be a morphism of presheaves on X . It is called a **local isomorphism** iff*

$$\mathcal{A}_x \longrightarrow \mathcal{B}_x$$

is an isomorphism for any $x \in X$.

Lemma 2.24. *Let*

$$\mathcal{A} \longrightarrow \mathcal{B}$$

be a morphism of precosheaves on X with values in $Pro(\mathbf{SET})$. Then it is a local isomorphism iff

$$(\kappa(\mathcal{B})(Z))_x \longrightarrow (\kappa(\mathcal{A})(Z))_x$$

is an isomorphism for any set Z and any $x \in X$.

Proof. Since κ is a full (contravariant) embedding,

$$\varphi : \mathcal{A}^x \longrightarrow \mathcal{B}^x$$

is an isomorphism iff

$$\kappa(\mathcal{B}^x) \xrightarrow{\kappa(\varphi)} \kappa(\mathcal{A}^x)$$

is an isomorphism. For any set Z , since κ converts cofiltrant limits to filtrant colimits, the composition

$$\begin{aligned} \kappa(\mathcal{B}^x)(Z) &\approx Hom_{Pro(\mathbf{SET})}(\mathcal{B}^x, Z) \approx Hom_{Pro(\mathbf{SET})}\left(\lim_{x \in U} \mathcal{B}(U), Z\right) \approx \\ &\approx colim_{x \in U} Hom_{Pro(\mathbf{SET})}(\mathcal{B}(U), Z) \approx (\kappa(\mathcal{B})(Z))_x \approx \\ &\approx (\kappa(\mathcal{B}))_x(Z) \longrightarrow (\kappa(\mathcal{A}))_x(Z) \approx \\ &\approx (\kappa(\mathcal{A})(Z))_x \approx colim_{x \in U} Hom_{Pro(\mathbf{SET})}(\mathcal{A}(U), Z) \approx \\ &\approx Hom_{Pro(\mathbf{SET})}\left(\lim_{x \in U} \mathcal{A}(U), Z\right) \approx Hom_{Pro(\mathbf{SET})}(\mathcal{A}^x, Z) \approx \kappa(\mathcal{A}^x)(Z) \end{aligned}$$

is an isomorphism iff $(\kappa(\mathcal{B})(Z))_x \longrightarrow (\kappa(\mathcal{A})(Z))_x$ is. \square

Lemma 2.25. *Let \mathcal{A} be a separated presheaf (e.g., a sheaf) of sets on X . Let a and b be two sections*

$$a, b \in \mathcal{A}(U).$$

Then $a = b$ iff $a|_x = b|_x$ for any $x \in U$.

Proof. For any $x \in U$, there exists an open neighborhood V_x , $x \in V_x \subseteq U$, such that

$$a|_{V_x} = b|_{V_x}.$$

Therefore, the images of a and b under the mapping

$$\xi : \mathcal{A}(U) \longrightarrow \prod_{x \in U} \mathcal{A}(V_x)$$

are equal. It follows that $a = b$ because ξ is a monomorphism. \square

Proposition 2.26. *Let*

$$\varphi : \mathcal{A} \longrightarrow \mathcal{B}$$

be a local isomorphism of sheaves of sets on X . Then φ is an isomorphism.

Proof. Let U be an open subset of X .

Step 1.

$$\varphi(U) : \mathcal{A}(U) \longrightarrow \mathcal{B}(U)$$

is onto. Indeed, let $b \in \mathcal{B}(U)$. For any $x \in U$, find a $a'(x) \in \mathcal{A}_x$ with

$$\varphi_x(a'(x)) = b(x) := b|_x.$$

There exist open neighborhoods V_x , $x \in V_x \subseteq U$, and sections $a(x) \in \mathcal{A}(V_x)$ with

$$\begin{aligned} a'(x) &= (a(x))|_x, \\ \varphi(a(x)) &= b|_{V_x}. \end{aligned}$$

Let now $x, y \in X$, $z \in V_x \cap V_y$. It follows that

$$\varphi_z((a(x))|_z) = b|_z = \varphi_z((a(y))|_z)$$

for any $z \in V_x \cap V_y$, therefore $(a(x))|_z = (a(y))|_z$ since φ_z are isomorphisms. Due to Lemma 2.25, the two sections are equal:

$$(a(x))|_{V_x \cap V_y} = (a(y))|_{V_x \cap V_y}$$

The (infinite) tuple $(a(x) : x \in X)$ lies in the kernel of

$$\prod_{x \in U} \mathcal{A}(V_x) \rightrightarrows \prod_{x, y \in U} \mathcal{A}(V_x \cap V_y).$$

Since \mathcal{A} is a sheaf, there exists a section $a \in \mathcal{A}(U)$ with $a|_{V_x} = a(x)$. Comparing the stalks $\varphi(a)|_x$ and $b|_x$ and using Lemma 2.25 again, conclude that $\varphi(a) = b$.

Step 2.

$$\varphi(U) : \mathcal{A}(U) \longrightarrow \mathcal{B}(U)$$

is one-to-one. Indeed, let $a_0, a_1 \in \mathcal{A}(U)$, and assume

$$\varphi(a_0) = \varphi(a_1).$$

Then for any $x \in U$,

$$\varphi_x((a_0)|_x) = \varphi_x((a_1)|_x)$$

and

$$(a_0)|_x = (a_1)|_x$$

since φ_x is one-to-one. Lemma 2.25 guarantees that $a_0 = a_1$. \square

Proposition 2.27. *Let*

$$\varphi : \mathcal{A} \longrightarrow \mathcal{B}$$

be a local isomorphism of cosheaves on X with values in $Pro(\mathbf{SET})$. Then φ is an isomorphism.

Proof. For any set Z , $\kappa(\mathcal{A})(Z)$ and $\kappa(\mathcal{B})(Z)$ are sheaves of sets. Lemma 2.24 guarantees that

$$\kappa(\varphi)(Z) : \kappa(\mathcal{B})(Z) \longrightarrow \kappa(\mathcal{A})(Z)$$

is a local isomorphism. Due to Proposition 2.26, $\kappa(\varphi)(Z)$ is an isomorphism of sheaves of sets, therefore

$$\kappa(\varphi) : \kappa(\mathcal{B}) \longrightarrow \kappa(\mathcal{A})$$

is an isomorphism, and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an isomorphism as well since κ is a full embedding. \square

3. PROOFS OF THE MAIN RESULTS

3.1. The plus-construction for precosheaves with values in $Pro(\mathbb{SET})$.

Definition 3.1. *Let*

$$\{U_i \longrightarrow U\} \in Cov(X),$$

and let \mathcal{A} be a precosheaf with values in $Pro(\mathbb{SET})$. Define

$$H_0(\{U_i \longrightarrow U\}, \mathcal{A}) := coker \left(\prod_{i,j} \mathcal{A}(U_i \times_U U_j) \rightrightarrows \prod_i \mathcal{A}(U_i) \right).$$

The definition is correct since $Pro(\mathbb{SET})$ is cocomplete.

Analogously, if \mathcal{B} is a presheaf with values in a complete category \mathbb{E} , define

$$H^0(\{U_i \longrightarrow U\}, \mathcal{B}) := \ker \left(\prod_i \mathcal{B}(U_i) \rightrightarrows \prod_{i,j} \mathcal{B}(U_i \cap U_j) \right).$$

Proposition 3.2.

$$\kappa(H_0(\{U_i \longrightarrow U\}, \mathcal{A})) = \ker \left(\prod_i \kappa(\mathcal{A}(U_i)) \rightrightarrows \prod_{i,j} \kappa(\mathcal{A}(U_i \times_U U_j)) \right) = H^0(\{U_i \longrightarrow U\}, \kappa(\mathcal{A})).$$

Proof. The functor κ converts colimits to limits. □

Definition 3.3. *Given two coverings*

$$\begin{aligned} \mathcal{V}, \mathcal{U} &\in Cov(X), \\ \mathcal{U} &= \{U_i \longrightarrow U\}_{i \in I}, \\ \mathcal{V} &= \{V_j \longrightarrow U\}_{j \in J}, \end{aligned}$$

*then a **refinement mapping** between them*

$$f : \mathcal{V} \longrightarrow \mathcal{U},$$

is a pair

$$\begin{aligned} \varepsilon &: J \longrightarrow I, \\ (f_j : V_j &\longrightarrow U_{\varepsilon(j)}), \end{aligned}$$

where f_j are U -morphisms.

Lemma 3.4. *Given two coverings*

$$\begin{aligned} \mathcal{V}, \mathcal{U} &\in Cov(X), \\ \mathcal{U} &= \{U_i \longrightarrow U\}_{i \in I}, \\ \mathcal{V} &= \{V_j \longrightarrow U\}_{j \in J}, \end{aligned}$$

and two refinement mappings

$$f, g : \mathcal{V} \longrightarrow \mathcal{U},$$

then the corresponding mappings of cokernels coincide:

$$H_0(f, \mathcal{A}) = H_0(g, \mathcal{A}) : H_0(\mathcal{V}, \mathcal{A}) \longrightarrow H_0(\mathcal{U}, \mathcal{A}).$$

Proof. Let Z be an arbitrary set. Then

$$\kappa(H_0(\mathcal{U}, \mathcal{A}))(Z) = H^0(\mathcal{U}, \kappa(\mathcal{A}))(Z)$$

where $\kappa(\mathcal{A})(Z)$ is a presheaf of sets. It follows from [Tam94], Lemma I.2.2.7, that

$$H^0(f, \kappa(\mathcal{A})) = H^0(g, \kappa(\mathcal{A})) : H^0(\mathcal{U}, \kappa(\mathcal{A})) \longrightarrow H^0(\mathcal{V}, \kappa(\mathcal{A})).$$

Therefore, $H_0(f, \mathcal{A}) = H_0(g, \mathcal{A})$ since κ is a full embedding. \square

Remark 3.5. In [Tam94] all the reasonings are done for presheaves of **abelian groups**. However, as the author underlines, the reasonings can be easily translated to the situation of presheaves of **sets**.

Definition 3.6. Given $U \in \text{Cat}(X)$, the set of coverings on U is a cofiltrant pre-ordered set under the refinement relation:

$$\mathcal{V} \leq \mathcal{U}$$

iff \mathcal{V} refines \mathcal{U} . Since the mappings

$$H_0(\mathcal{V}, \mathcal{A}) \longrightarrow H_0(\mathcal{U}, \mathcal{A})$$

do not depend on the refinement mapping (Lemma 3.4), and since $\text{Pro}(\text{SET})$ admits cofiltrant limits, one can define

$$\mathcal{A}_+(U) := \lim_{\mathcal{V}} H_0(\mathcal{V}, \mathcal{A})$$

where \mathcal{V} runs over coverings on U . \mathcal{A}_+ is clearly a precosheaf in $\text{Pro}(\text{SET})$.

Definition 3.7. Given a presheaf \mathcal{B} on X with values in SET or SET^{SET} , let \mathcal{B}^+ be the following presheaf:

$$\mathcal{B}^+(U) := \text{colim}_{\mathcal{V}} H^0(\mathcal{V}, \mathcal{B}).$$

Proposition 3.8.

$$\kappa(\mathcal{A}_+) \approx (\kappa(\mathcal{A}))^+.$$

Proof. Follows from Propositions 3.2 and 4.12. \square

Proposition 3.9. (1) \mathcal{A}_+ is coseparated.

(2) If \mathcal{A} is coseparated, then \mathcal{A}_+ is a cosheaf.

(3) The functor

$$()_{\#} := ()_{++} : \text{PCS}(X, \text{Pro}(\text{SET})) \longrightarrow \text{CS}(X, \text{Pro}(\text{SET}))$$

is right adjoint to the inclusion functor.

Proof.

(1) Since

$$\kappa(\mathcal{A}_+) \approx (\kappa(\mathcal{A}))^+$$

is separated ([Tam94], Proposition I.3.1.3), it follows from Proposition 2.11 that \mathcal{A}_+ is coseparated.

(2) If \mathcal{A} is coseparated, then $\kappa(\mathcal{A})$ is separated, therefore $\kappa(\mathcal{A}_+) \approx (\kappa(\mathcal{A}))^+$ is a sheaf ([Tam94], Proposition I.3.1.3). It follows from Proposition 2.12 that \mathcal{A}_+ is a cosheaf.

(3) For a fixed set Z , the functor

$$B(Z) \mapsto B(Z)^{++} = B(Z)^\# : \mathcal{PS}(X, \mathcal{SET}) \longrightarrow \mathcal{S}(X, \mathcal{SET}).$$

is left adjoint to the inclusion functor

$$I : \mathcal{S}(X, \mathcal{SET}) \longrightarrow \mathcal{PS}(X, \mathcal{SET})$$

([Tam94], Proposition I.3.1.3). Varying Z , one gets a pair of adjoint functors

$$\begin{aligned} I & : \mathcal{S}(X, \mathcal{SET}^{\mathcal{SET}}) \longrightarrow \mathcal{PS}(X, \mathcal{SET}^{\mathcal{SET}}), \\ ()^\# & : \mathcal{PS}(X, \mathcal{SET}^{\mathcal{SET}}) \longrightarrow \mathcal{S}(X, \mathcal{SET}^{\mathcal{SET}}). \end{aligned}$$

Let \mathcal{A} be a cosheaf, and \mathcal{B} be a precosheaf. There is a sequence of natural isomorphisms:

$$\begin{aligned} \text{Hom}_{\mathcal{CS}(X, \text{Pro}(\mathcal{SET}))}(\mathcal{A}, \mathcal{B}_\#) & \approx \text{Hom}_{\mathcal{S}(X, \mathcal{SET}^{\mathcal{SET}})}(\kappa(\mathcal{B}_\#), \kappa(\mathcal{A})) \\ & \approx \text{Hom}_{\mathcal{S}(X, \mathcal{SET}^{\mathcal{SET}})}(\kappa(\mathcal{B})^\#, \kappa(\mathcal{A})) \\ & \approx \text{Hom}_{\mathcal{PS}(X, \mathcal{SET}^{\mathcal{SET}})}(\kappa(\mathcal{B}), \kappa(\mathcal{A})) \\ & \approx \text{Hom}_{\mathcal{PCS}(X, \text{Pro}(\mathcal{SET}))}(\mathcal{A}, \mathcal{B}) \end{aligned}$$

establishing the desired adjunction. \square

3.2. The plus-construction for precosheaves with values in $\text{Pro}(\mathbb{A}\mathbb{B})$. We translate the definitions and statements of Subsection 3.1 from the language of pro-sets into the language of abelian pro-groups. The proofs are omitted since they are completely analogous to the corresponding proofs for pro-sets.

Remark 3.10. The symbol \coprod in this Subsection will mean the coproduct in the category $\text{Pro}(\mathbb{A}\mathbb{B})$. We could use the symbol \bigoplus instead, but this would have been inconsistent with the notations from the previous Subsection, where \coprod means the coproduct in the category $\text{Pro}(\mathcal{SET})$.

Definition 3.11. Let

$$\mathcal{U} = \{U_i \longrightarrow U\} \in \text{Cov}(X)$$

and \mathcal{A} a precosheaf with values in $\text{Pro}(\mathbb{A}\mathbb{B})$. Define

$$H_0(\mathcal{U}, \mathcal{A}) = \text{coker} \left(\coprod_{i,j} \mathcal{A}(U_i \times_U U_j) \rightrightarrows \coprod_i \mathcal{A}(U_i) \right).$$

The definition is correct since $\text{Pro}(\mathbb{A}\mathbb{B})$ is cocomplete.

Proposition 3.12.

$$\kappa(H_0(\mathcal{U}, \mathcal{A})) = \ker \left(\prod_i \kappa(\mathcal{A}(U_i)) \rightrightarrows \prod_{i,j} \kappa(\mathcal{A}(U_i \times_U U_j)) \right) = H^0(\mathcal{U}, \kappa(\mathcal{A})).$$

Lemma 3.13. *Given two coverings*

$$\mathcal{V}, \mathcal{U} \in \text{Cov}(X)$$

and two refinement mappings

$$f, g : \mathcal{V} \longrightarrow \mathcal{U},$$

then the corresponding mapping of cokernels coincide:

$$H_0(f, \mathcal{A}) = H_0(g, \mathcal{A}) : H_0(\mathcal{V}, \mathcal{A}) \longrightarrow H_0(\mathcal{U}, \mathcal{A}).$$

Definition 3.14. *Given $U \in \text{Cat}(X)$, the set of coverings on U is a cofiltrant pre-ordered set under the refinement relation:*

$$\mathcal{V} \leq \mathcal{U}$$

iff \mathcal{V} refines \mathcal{U} . Since the mappings

$$H_0(\mathcal{V}, \mathcal{A}) \longrightarrow H_0(\mathcal{U}, \mathcal{A})$$

do not depend on the refinement mapping (Lemma 3.13), and since $\text{Pro}(\mathbb{A}\mathbb{B})$ admits cofiltrant limits, one can define

$$\mathcal{A}_+(U) := \lim_{\mathcal{V}} H_0(\mathcal{V}, \mathcal{A})$$

where \mathcal{V} runs over open coverings on U . \mathcal{A}_+ is clearly a precosheaf with values in $\text{Pro}(\mathbb{A}\mathbb{B})$.

Definition 3.15. *Given a presheaf \mathcal{B} on X , let \mathcal{B}^+ be the following presheaf:*

$$\mathcal{B}^+(U) := \text{colim}_{\mathcal{V}} H^0(\mathcal{V}, \mathcal{B}).$$

Proposition 3.16.

$$\kappa(\mathcal{A}_+) \approx (\kappa(\mathcal{A}))^+.$$

Proposition 3.17. (1) \mathcal{A}_+ is coseparated.

(2) If \mathcal{A} is coseparated, then \mathcal{A}_+ is a cosheaf.

(3) The functor

$$()_{\#} := ()_{++} : \text{PCS}(\text{Pro}(\mathbb{A}\mathbb{B})) \longrightarrow \text{CS}\left(\text{Pro}\left(\mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}\right)\right)$$

is right adjoint to the inclusion functor.

3.3. Proof of Theorems 1.1 and 1.2.

Proof. The right adjointness of the functors

$$()_{\#} : \text{PCS}(X, \text{Pro}(\text{SET})) \longrightarrow \text{CS}(X, \text{Pro}(\text{SET}))$$

and

$$()_{\#} : \text{PCS}(X, \text{Pro}(\mathbb{A}\mathbb{B})) \longrightarrow \text{CS}(X, \text{Pro}(\mathbb{A}\mathbb{B}))$$

is already proven (Propositions 3.9 and 3.17). It remains only to prove that $()_{\#}$ is a reflector. Let \mathbb{D} be $\text{Pro}(\text{SET})$ (or $\text{Pro}(\mathbb{A}\mathbb{B})$), and let \mathcal{A} be a cosheaf with values in \mathbb{D} . Let further \mathcal{B} be an arbitrary cosheaf with values in \mathbb{D} . It is enough to prove that

$$\text{Hom}_{\text{PCS}(X, \mathbb{D})}(\mathcal{B}, \mathcal{A}_{\#}) \approx \text{Hom}_{\text{CS}(X, \mathbb{D})}(\mathcal{B}, \mathcal{A})$$

naturally on \mathcal{B} . There exist natural on \mathcal{B} isomorphisms

$$\begin{aligned} \text{Hom}_{\text{PCS}(X, \mathbb{D})}(\mathcal{B}, \mathcal{A}_{\#}) &\approx \text{Hom}_{\text{CS}(X, \mathbb{D})}(\mathcal{B}, \mathcal{A}_{\#}) \approx \\ \text{Hom}_{\text{PCS}(X, \mathbb{D})}(\mathcal{B}, \mathcal{A}) &\approx \text{Hom}_{\text{CS}(X, \mathbb{D})}(\mathcal{B}, \mathcal{A}). \end{aligned}$$

The first and the last isomorphisms are due to the full embedding of $\mathbb{CS}(X, \mathbb{D})$ into $\mathbb{PCS}(X, \mathbb{D})$, while the second isomorphism is the adjunction. It follows that

$$\mathcal{A}_{\#} \approx \mathcal{A}.$$

□

Remark 3.18. *The reasoning above can be easily generalized to any full embedding*

$$I : \mathbb{E} \longrightarrow \mathbb{F}.$$

If such an embedding has a right or a left adjoint F , then F is clearly a reflector.

3.4. Proof of Theorems 1.3 and 1.4.

Proof. (2) is already proven (Proposition 2.27).

To prove (1), consider an arbitrary **set** Z . Corollary 3.20 below guarantees that

$$\beta^{\#}(\kappa(\mathcal{A}))(Z) = \kappa(\alpha_{\#}(\mathcal{A}))(Z) : \kappa(\mathcal{A})(Z) \longrightarrow \kappa(\mathcal{A}_{\#})(Z) = \kappa(\mathcal{A})^{\#}(Z)$$

is a local isomorphism for any set Z . Apply now Lemma 2.24. The case of pre-cosheaves with values in $Pro(\mathbb{A}\mathbb{B})$ is proved analogously (consider an **abelian group** Z instead).

To prove (3), consider the composition

$$\mathcal{B} \longrightarrow \mathcal{A}_{\#} \longrightarrow \mathcal{A},$$

existing due to the right adjointness of $()_{\#}$. The composition and the morphism $\mathcal{A}_{\#} \longrightarrow \mathcal{A}$ are local isomorphisms, therefore $\mathcal{B} \longrightarrow \mathcal{A}_{\#}$ is a local isomorphism between cosheaves, hence an isomorphism. □

Lemma 3.19. *Let X be a topological space. Given a presheaf \mathcal{B} of sets on X , the natural morphism*

$$\beta^+(\mathcal{B}) : \mathcal{B} \longrightarrow \mathcal{B}^+$$

is a local isomorphism.

Proof. Step 1. For any $x \in X$, $\beta^+(\mathcal{B})_x$ is onto. Indeed, let the equivalence class

$$[(s(i))_{i \in I}] \in (\mathcal{B}^+)_x$$

be given by sections

$$\begin{aligned} s(i) &\in \mathcal{B}(U_i), \\ s(i)|_{U_i \cap U_j} &= s(j)|_{U_i \cap U_j}, \end{aligned}$$

where V is an open neighborhood of x , and $\{U_i \longrightarrow V\}_{i \in I}$ is an open covering. There exists a j with $x \in U_j$. The class

$$[(s(j), U_j)] \in \mathcal{B}_x$$

is clearly mapped onto $[(s(i))_{i \in I}]$ under the mapping $\beta^+(\mathcal{B})_x$.

Step 2. For any $x \in X$, $\beta^+(\mathcal{B})_x$ is one-to-one. Indeed, let

$$[s], [t] \in \mathcal{B}_x,$$

where $s, t \in \mathcal{B}(V)$ are sections, and V is an open neighborhood of x . Assume

$$\beta^+(\mathcal{B})_x([s]) = \beta^+(\mathcal{B})_x([t]).$$

This means that there exists an open neighborhood W , $x \in W \subseteq V$, and an open covering $\{U_i \longrightarrow W\}$ with

$$s|_{U_i} = t|_{U_i}$$

for all i . There exists then a j with $x \in U_j$. Since $s|_{U_j} = t|_{U_j}$, the classes $[s]$ and $[t]$ are equal in \mathcal{B}_x . \square

Corollary 3.20. *Let X be a topological space. Given a presheaf \mathcal{B} of sets on X , the natural morphism*

$$\beta^\#(\mathcal{B}) : \mathcal{B} \longrightarrow \mathcal{B}^\#$$

is a local isomorphism.

Proof. $\beta^\#(\mathcal{B}) = \beta^+(\mathcal{B}^+) \circ \beta^+(\mathcal{B})$. \square

3.5. Proof of Theorem 1.7.

Proof. (1) Let Z be a set. Then, due to Lemma 3.21 below, the presheaf

$$\kappa(\mathcal{P})(Z) = \kappa(S \times \text{pro-}\pi_0)(Z)$$

of sets is isomorphic to the presheaf \mathcal{B} :

$$\mathcal{B}(U) := Z^{S \times U}.$$

For any open covering $\{U_i \longrightarrow U\}$ the topological space $S \times U$ (S with the discrete topology) is isomorphic in the category \mathbb{TOP} to the cokernel

$$\text{coker} \left(\coprod_{i,j} (S \times (U_i \cap U_j)) \rightrightarrows \coprod_{i,j} (S \times U_i) \right),$$

therefore

$$\begin{aligned} Z^{S \times U} &= \text{Hom}_{\mathbb{TOP}}(S \times U, Z) \approx \\ &\approx \ker \left(\text{Hom}_{\mathbb{TOP}} \left(\coprod_{i,j} (S \times U_i), Z \right) \rightrightarrows \text{Hom}_{\mathbb{TOP}} \left(\coprod_{i,j} (S \times (U_i \cap U_j)), Z \right) \right) \approx \\ &\approx \ker \left(\prod_i \mathcal{B}(U_i) \rightrightarrows \prod_{i,j} \mathcal{B}(U_i \cap U_j) \right), \end{aligned}$$

and \mathcal{B} is a sheaf of sets. It follows from Proposition 2.12 that $\mathcal{P} = S \times \text{pro-}\pi_0$ is a cosheaf.

(2) It is enough to prove that

$$\mathcal{P} = S \times \text{pro-}\pi_0 \longrightarrow S^{LC}$$

is a local isomorphism. Let Z be a set, and let $x \in X$. Clearly

$$\kappa(S^{LC})(Z)_x \approx Z^S.$$

Moreover,

$$(\kappa(\mathcal{P}))_x = (\kappa(S \times \text{pro-}\pi_0)(Z))_x = \text{colim}_{x \in V} Z^{S \times V}$$

where the colimit is taken over all open neighborhoods V of x . The mappings

$$S \times V \longrightarrow Z$$

involved are locally constant since Z is discrete. Therefore, any two germs $[f]$ and $[g]$,

$$f, g : W \longrightarrow Z$$

where W is an open neighborhood of $S \times \{x\}$, are equivalent iff

$$f|_{S \times \{x\}} = g|_{S \times \{x\}}.$$

It follows that

$$\operatorname{colim}_{x \in V} Z^{S \times V} \approx Z^S,$$

and that both the mapping of presheaves

$$\kappa(S^{LC}) \longrightarrow \kappa(S \times \operatorname{pro}\pi_0)(Z)$$

and the mapping of precosheaves

$$\mathcal{P} = S \times \operatorname{pro}\pi_0 \longrightarrow S^{LC}$$

are local isomorphisms. Due to Theorem 1.3,

$$\mathcal{P} = S \times \operatorname{pro}\pi_0 \approx (S^{LC})_{\#}.$$

(3) Let Z be an abelian group. Then, due to Lemma 3.22 below, the presheaf

$$\kappa(\operatorname{pro}\text{-}H_0(-, A))(Z)$$

of abelian groups is isomorphic to the presheaf \mathcal{C} :

$$\mathcal{C}(U) := \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)^U.$$

For any open covering $\{U_i \longrightarrow U\}$ the topological space U is isomorphic in the category \mathbf{TOP} to the cokernel

$$\operatorname{coker} \left(\coprod_{i,j} (U_i \cap U_j) \rightrightarrows \coprod_{i,j} U_i \right),$$

therefore

$$\begin{aligned} \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)^U &= \operatorname{Hom}_{\mathbf{TOP}}(U, \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)) \approx \\ &\approx \ker \left(\operatorname{Hom}_{\mathbf{TOP}} \left(\coprod_{i,j} U_i, \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z) \right) \rightrightarrows \operatorname{Hom}_{\mathbf{TOP}} \left(\coprod_{i,j} (U_i \cap U_j), \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z) \right) \right) \approx \\ &\approx \ker \left(\prod_i \mathcal{C}(U_i) \rightrightarrows \prod_{i,j} \mathcal{C}(U_i \cap U_j) \right), \end{aligned}$$

and \mathcal{C} is a sheaf of sets. It follows from Proposition 2.15 that $\mathcal{H} = \operatorname{pro}\text{-}H_0(-, A)$ is a cosheaf.

(4) It is enough to prove that

$$\mathcal{H} = \operatorname{pro}\text{-}H_0(-, A) \longrightarrow A^{LC}$$

is a local isomorphism. Let Z be an abelian group, and let $x \in X$. Clearly

$$\kappa(A^{LC})(Z)_x \approx \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z).$$

Moreover,

$$(\kappa(\mathcal{H}))_x = (\kappa(\operatorname{pro}\text{-}H_0(-, A))(Z))_x = \operatorname{colim}_{x \in V} \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)^V$$

where the colimit is taken over all open neighborhoods V of x . The mappings

$$V \longrightarrow \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)$$

involved are locally constant since $\operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)$ is discrete. Therefore, any two germs $[f]$ and $[g]$,

$$f, g : W \longrightarrow \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)$$

where W is an open neighborhood of x , are equivalent iff

$$f(x) = g(x).$$

It follows that

$$\operatorname{colim}_{x \in V} \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)^V \approx \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z),$$

and that both the mapping of presheaves

$$\kappa(A^{LC}) \longrightarrow \kappa(\operatorname{pro}\text{-}H_0(-, A))(Z) = \kappa(\mathcal{H})(Z)$$

and the mapping of precosheaves

$$\mathcal{H} = \operatorname{pro}\text{-}H_0(-, A) \longrightarrow A^{LC}$$

are local isomorphisms. Due to Theorem 1.4,

$$\mathcal{H} = \operatorname{pro}\text{-}H_0(-, A) \approx (A^{LC})_{\#}.$$

□

Lemma 3.21. *For any set Z and any topological space U , the set*

$$\operatorname{Hom}_{\mathbf{SET}}(S \times \operatorname{pro}\text{-}\pi_0(U), Z)$$

is naturally (with respect to S , Z and U) isomorphic to the set $Z^{S \times U}$ of continuous functions

$$S \times U \longrightarrow Z$$

where S and Z are supplied with the discrete topology.

Proof. Let $U \longrightarrow (Y_j)$ be a strong expansion ([Mar00], conditions (S1) and (S2) on p. 129), where (Y_j) is a pro-space consisting of polyhedra (or ANRs), and let

$$S \times \operatorname{pro}\text{-}\pi_0(Y_j) = (S \times \pi_0(Y_j))$$

be the corresponding pro-set. Since the spaces Y_j are locally path-connected, and since Z is a discrete topological space, one has a sequence of isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\operatorname{Pro}(\mathbf{SET})}((S \times \operatorname{pro}\text{-}\pi_0(Y_j)), Z) &\approx \operatorname{Hom}_{\operatorname{Pro}(\mathbf{SET})}((S \times \pi_0(Y_j)), Z) \approx \\ &\approx \operatorname{colim}_j \operatorname{Hom}_{\mathbf{SET}}(S \times \pi_0(Y_j), Z) \approx \\ &\approx \operatorname{colim}_j \operatorname{Hom}_{\mathbf{TOP}}(S \times Y_j, Z). \end{aligned}$$

The compositions

$$S \times U \longrightarrow S \times Y_j \longrightarrow Z$$

define a natural mapping

$$\operatorname{colim}_j [S \times Y_j, Z] \longrightarrow [S \times U, Z]$$

where $[]$ is the set of homotopy classes of mappings. That mapping is an isomorphism because $U \longrightarrow (Y_j)$ is a strong expansion. Since Z is discrete, the homotopy classes of mappings

$$S \times U \longrightarrow Z$$

and

$$S \times Y_j \longrightarrow Z$$

consist of single mappings, therefore

$$\operatorname{colim}_j [S \times Y_j, Z] = \operatorname{colim}_j \operatorname{Hom}_{\mathbf{TOP}}(S \times Y_j, Z) \longrightarrow \operatorname{Hom}_{\mathbf{TOP}}(S \times U, Z) = [S \times U, Z],$$

and the internal mapping

$$Hom_{Pro(\mathbf{SET})}((S \times \varinjlim_j pro-\pi_0(Y_j)), Z) \approx colim_j Hom_{\mathbf{TOP}}(S \times Y_j, Z) \longrightarrow Hom_{\mathbf{TOP}}(S \times U, Z)$$

is an isomorphism. \square

Lemma 3.22. *For any abelian group Z and any topological space U , the set*

$$Hom_{\mathbb{A}\mathbb{B}}(pro-H_0(U, A), Z)$$

is naturally (with respect to A , Z and U) isomorphic to the Čech cohomology group

$$\check{H}^0(U, Hom_{\mathbb{A}\mathbb{B}}(A, Z))$$

which, in turn, is isomorphic to the group $Hom_{\mathbb{A}\mathbb{B}}(A, Z)^U$ of continuous functions

$$U \longrightarrow Hom_{\mathbb{A}\mathbb{B}}(A, Z)$$

where $Hom_{\mathbb{A}\mathbb{B}}(A, Z)$ is supplied with the discrete topology.

Proof. Let again $U \longrightarrow (Y_j)$ be a polyhedral (or ANR) strong expansion, and let

$$pro-H_0(Y_j, A) = (H_0(Y_j, A))$$

be the corresponding abelian pro-group. Since the spaces Y_j are locally path-connected,

$$H_0(Y_j, A) = \bigoplus_{\pi_0(Y_j)} A$$

and

$$Hom_{\mathbf{TOP}}(Y_j, V) = Hom_{\mathbf{SET}}(\pi_0(Y_j), V)$$

for any discrete topological space V . Since $Hom_{\mathbb{A}\mathbb{B}}(A, Z)$ is considered as a discrete topological space, one has a sequence of isomorphisms

$$\begin{aligned} Hom_{Pro(\mathbb{A}\mathbb{B})}((H_0(Y_j, A)), Z) &\approx colim_j Hom_{\mathbb{A}\mathbb{B}}\left(\left(\bigoplus_{\pi_0(Y_j)} A\right), Z\right) \approx \\ &\approx colim_j \prod_{\pi_0(Y_j)} Hom_{\mathbb{A}\mathbb{B}}(A, Z) \approx \\ &\approx colim_j Hom_{\mathbf{SET}}(\pi_0(Y_j), Hom_{\mathbb{A}\mathbb{B}}(A, Z)) \approx \\ &\approx colim_j Hom_{\mathbf{TOP}}(Y_j, Hom_{\mathbb{A}\mathbb{B}}(A, Z)). \end{aligned}$$

The compositions

$$U \longrightarrow Y_j \longrightarrow Hom_{\mathbb{A}\mathbb{B}}(A, Z)$$

define a natural mapping

$$colim_j [Y_j, Hom_{\mathbb{A}\mathbb{B}}(A, Z)] \longrightarrow [U, Hom_{\mathbb{A}\mathbb{B}}(A, Z)].$$

That mapping is an isomorphism because $U \longrightarrow (Y_j)$ is a strong expansion. Since $Hom_{\mathbb{A}\mathbb{B}}(A, Z)$ is discrete, the homotopy classes of mappings

$$U \longrightarrow Hom_{\mathbb{A}\mathbb{B}}(A, Z)$$

and

$$Y_j \longrightarrow Hom_{\mathbb{A}\mathbb{B}}(A, Z)$$

consist of single mappings, therefore

$$\begin{aligned} \operatorname{colim}_j [Y_j, \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)] &= \operatorname{colim}_j \operatorname{Hom}_{\operatorname{TOP}}(Y_j, \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)) \longrightarrow \\ &\longrightarrow \operatorname{Hom}_{\operatorname{TOP}}(U, \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)) = [U, \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)], \end{aligned}$$

and the internal mapping

$$\operatorname{Hom}_{\operatorname{Pro}(\mathbb{A}\mathbb{B})}((H_0(Y_j, A)), Z) \approx \operatorname{colim}_j \operatorname{Hom}_{\operatorname{TOP}}(Y_j, \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)) \longrightarrow \operatorname{Hom}_{\operatorname{TOP}}(U, \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z))$$

is an isomorphism. The Čech cohomology group

$$\check{H}^0(U, \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z))$$

is isomorphic to

$$\begin{aligned} \operatorname{Hom}_{\operatorname{Pro}(\mathbb{A}\mathbb{B})}((H_0(Y_j)), \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)) &\approx \operatorname{colim}_j \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(H_0(Y_j), \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(A, Z)) \approx \\ &\approx \operatorname{colim}_j \operatorname{Hom}_{\mathbb{A}\mathbb{B}}(H_0(Y_j) \otimes A, Z) \approx \\ &\approx \operatorname{colim}_j \operatorname{Hom}_{\mathbb{A}\mathbb{B}}\left(\left(\bigoplus_{\pi_0(Y_j)} A\right), Z\right) \approx \\ &\approx \operatorname{Hom}_{\operatorname{Pro}(\mathbb{A}\mathbb{B})}((H_0(Y_j, A)), Z). \end{aligned}$$

□

4. APPENDIX: CATEGORIES $\operatorname{Pro}(\operatorname{SET})$ AND $\operatorname{Pro}(\mathbb{A}\mathbb{B})$

Let us remind necessary notions from category theory.

We fix a **universe** \mathfrak{U} ([KS06], Definition 1.1.1).

Definition 4.1. A set is called **small** (\mathfrak{U} -small in the terminology of [KS06], Definition 1.1.2) if it is isomorphic to a set belonging to \mathfrak{U} . A category \mathbb{D} is called **small** if both the set of objects $\operatorname{Ob}(\mathbb{D})$ and the set of morphisms $\operatorname{Mor}(\mathbb{D})$ are small.

Definition 4.2. A category \mathbb{D} is called a \mathfrak{U} -category ([KS06], Definition 1.2.1) if

$$\operatorname{Hom}_{\mathbb{D}}(X, Y)$$

is small for any two objects X and Y .

Definition 4.3. A **small limit** (**small colimit**) in a category \mathbb{D} is a limit (colimit) of a diagram

$$\Delta : I \longrightarrow \mathbb{D}$$

where I is a small category.

Definition 4.4. A **filtrant** (**cofiltrant**) **colimit** (**limit**) in a category \mathbb{D} is a colimit (limit) of a diagram

$$\Delta : I \longrightarrow \mathbb{D}$$

where I is a small filtrant (cofiltrant) category ([KS06], Definition 3.1.1).

Definition 4.5. A category \mathbb{D} is called **complete** (**cocomplete**) if \mathbb{D} admits small limits (colimits).

Definition 4.6. We denote by SET the category of small sets, and by $\mathbb{A}\mathbb{B}$ the (additive) category of small abelian groups. These two categories are clearly \mathfrak{U} -categories.

Definition 4.7. For a category \mathbb{D} , let $\mathbf{SET}^{\mathbb{D}}$ be the category of functors

$$\mathbb{D} \longrightarrow \mathbf{SET}.$$

For an **additive** category \mathbb{D} , let $\mathbf{AB}^{\mathbb{D}}$ be the category of **additive** functors

$$\mathbb{D} \longrightarrow \mathbf{AB}.$$

These two categories are in general **not** \mathcal{U} -categories (unless \mathbb{D} is a small category).

4.1. *Pro*(\mathbf{SET}). For a category \mathbb{D} , let

$$\iota : \mathbb{D}^{op} \longrightarrow \mathbf{SET}^{\mathbb{D}}$$

be the Ioneda full embedding:

$$\iota(X) := \text{Hom}_{\mathbb{D}}(X, -) : \mathbb{D} \longrightarrow \mathbf{SET},$$

and let

$$\kappa : \mathbb{D} \longrightarrow \mathbf{SET}^{\mathbb{D}}$$

be the corresponding contravariant embedding.

Definition 4.8. ([KS06], Definition 6.1.1) The category *Pro*(\mathbb{D}) is the opposite category $(\mathbb{E})^{op}$ where

$$\mathbb{E} \subseteq \mathbf{SET}^{\mathbb{D}}$$

is the full subcategory of functors that are filtrant colimits of representable functors, i.e. colimits of diagrams of the form

$$I^{op} \xrightarrow{X^{op}} \mathbb{D}^{op} \xrightarrow{\iota} \mathbf{SET}^{\mathbb{D}}$$

where I^{op} is a small filtrant category, and

$$X : I \longrightarrow \mathbb{D}$$

is a functor.

For simplicity, denote two such diagrams (and corresponding pro-objects) by $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$. Then

$$\text{Hom}_{\text{Pro}(\mathbb{D})}((X_i)_{i \in I}, (Y_j)_{j \in J}) = \lim_j \text{colim}_i \text{Hom}_{\mathbb{D}}(X_i, Y_j).$$

Definition 4.9. The full embedding

$$\iota : (\text{Pro}(\mathbf{SET}))^{op} \longrightarrow \mathbf{SET}^{\mathbf{SET}}$$

will be also called the Ioneda embedding, and will be denoted by the same symbol ι .

Definition 4.10. Let us denote by κ the corresponding contravariant embedding

$$\kappa : \text{Pro}(\mathbf{SET}) \longrightarrow \mathbf{SET}^{\mathbf{SET}}.$$

Proposition 4.11. The Ioneda embedding

$$\iota : (\text{Pro}(\mathbf{SET}))^{op} \longrightarrow \mathbf{SET}^{\mathbf{SET}}$$

commutes with small limits and filtrant colimits.

A morphism

$$f : X \longrightarrow Y$$

in $(\text{Pro}(\mathbf{SET}))^{op}$ is a monomorphism iff $\iota(f)$ is a monomorphism in $\mathbf{SET}^{\mathbf{SET}}$.

Proof. [KS06], Theorem 6.1.8, applied to the category \mathbf{SET}^{op} , states that the natural embedding

$$\iota : (Pro(\mathbf{SET}))^{op} \approx Ind(\mathbf{SET}^{op}) \longrightarrow \mathbf{SET}^{\mathbf{SET}}$$

commutes with filtrant colimits. Corollary 6.1.17 says that ι commutes with small limits. The last sentence of the Proposition reduces to (3) in Proposition 4.12, because

$$f^{op} : X \longrightarrow Y$$

is a monomorphism in $(Pro(\mathbf{SET}))^{op}$ iff f is an epimorphism in $Pro(\mathbf{SET})$. Moreover, $\iota(f^{op}) = \kappa(f)$. \square

Proposition 4.12. (1) *The **contravariant** embedding*

$$\kappa : Pro(\mathbf{SET}) \longrightarrow \mathbf{SET}^{\mathbf{SET}}$$

converts small colimits in $Pro(\mathbf{SET})$ to limits in $\mathbf{SET}^{\mathbf{SET}}$. Moreover, morphisms

$$(X_i \longrightarrow X)_{i \in I},$$

where I is a small category, form a colimit in $Pro(\mathbf{SET})$ iff

$$(\kappa(X) \longrightarrow \kappa(X_i))_{i \in I}$$

form a limit in $\mathbf{SET}^{\mathbf{SET}}$.

(2) *The embedding κ converts cofiltrant limits in $Pro(\mathbf{SET})$ to filtrant colimits in $\mathbf{SET}^{\mathbf{SET}}$. Moreover, given a small cofiltrant diagram $(X_i)_{i \in I}$, then morphisms*

$$(X \longrightarrow X_i)_{i \in I}$$

form a limit in $Pro(\mathbf{SET})$ iff

$$(\kappa(X_i) \longrightarrow \kappa(X))_{i \in I}$$

form a colimit in $\mathbf{SET}^{\mathbf{SET}}$.

(3) *A morphism*

$$f : X \longrightarrow Y$$

in $Pro(\mathbf{SET})$ is an epimorphism iff $\kappa(f)$ is a monomorphism in $\mathbf{SET}^{\mathbf{SET}}$.

Proof. All the statements are proved in [KS06], Part 6. Let us sketch the proofs here.

(1) Assume $Y = (Y_j)_{j \in J}$ is a pro-set. Then

$$\begin{aligned} \lim_i Hom_{Pro(\mathbf{SET})}(X_i, (Y_j)) &\approx \lim_i \lim_j Hom_{Pro(\mathbf{SET})}(X_i, Y_j) \approx \\ &\approx \lim_i \lim_j \kappa(X_i)(Y_j) \approx \lim_j \lim_i \kappa(X_i)(Y_j) \approx \\ &\approx \lim_j Hom_{Pro(\mathbf{SET})}\left(\lim_i X_i, Y_j\right). \end{aligned}$$

The sets $\lim_j Hom_{Pro(\mathbf{SET})}\left(\lim_i X_i, Y_j\right)$ are isomorphic to $\lim_j Hom_{Pro(\mathbf{SET})}(X, Y_j)$ iff $\kappa(X)$ is the limit of $(\kappa(X_i))_{i \in I}$.

(2) Follows from the fact that $Pro(\mathbf{SET})$ admits cofiltrant limits, and κ converts such limits to filtrant colimits ([KS06], Theorem 6.1.8).

(3) Assume $Z = (Z_j)_{j \in J}$ is a pro-set. Then the composition

$$\begin{aligned} \text{Hom}_{\text{Pro}(\text{SET})}(Y, Z) &\approx \lim_j \text{Hom}_{\text{Pro}(\text{SET})}(Y, (Z_j)) \approx \\ &\approx \lim_j \kappa(Y)(Z_j) \longrightarrow \lim_j \kappa(X)(Z_j) \approx \\ &\approx \lim_j \text{Hom}_{\text{Pro}(\text{SET})}(Y, (Z_j)) \approx \text{Hom}_{\text{Pro}(\text{SET})}(Y, Z) \end{aligned}$$

is a monomorphism whenever $\kappa(Y) \longrightarrow \kappa(X)$ is since \lim converts monomorphisms to monomorphisms. \square

4.2. *Pro*($\mathbb{A}\mathbb{B}$). The proofs in this Subsection are analogous to those in Subsection 4.1, and will be omitted.

For an additive category \mathbb{D} , let

$$\iota : \mathbb{D}^{op} \longrightarrow \mathbb{A}\mathbb{B}^{\mathbb{D}}$$

be the Ioneda full embedding:

$$\iota(X) := \text{Hom}_{\mathbb{D}}(X, _) : \mathbb{D} \longrightarrow \mathbb{A}\mathbb{B},$$

and let

$$\kappa : \mathbb{D} \longrightarrow \mathbb{A}\mathbb{B}^{\mathbb{D}}$$

be the corresponding contravariant embedding.

Definition 4.13. ([KS06], Part 15) *The category $\text{Pro}(\mathbb{D})$ is the opposite category $(\mathbb{E})^{op}$ where*

$$\mathbb{E} \subseteq \mathbb{A}\mathbb{B}^{\mathbb{D}}$$

is the full subcategory of additive functors that are filtrant colimits of representable functors, i.e. colimits of diagrams of the form

$$I^{op} \xrightarrow{X^{op}} \mathbb{D}^{op} \xrightarrow{\iota} \mathbb{A}\mathbb{B}^{\mathbb{D}}$$

where I^{op} is a small filtrant category, and

$$X : I \longrightarrow \mathbb{D}$$

is a functor.

For simplicity, denote two such diagrams (and corresponding pro-objects) by $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$. Then

$$\text{Hom}_{\text{Pro}(\mathbb{D})}\left((X_i)_{i \in I}, (Y_j)_{j \in J}\right) = \lim_j \text{colim}_i \text{Hom}_{\mathbb{D}}(X_i, Y_j).$$

Definition 4.14. *The full embedding*

$$\iota : (\text{Pro}(\mathbb{A}\mathbb{B}))^{op} \longrightarrow \mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$$

will be also called the Ioneda embedding, and will be denoted by the same symbol ι .

Definition 4.15. *Let us denote by κ the corresponding contravariant embedding*

$$\kappa : \text{Pro}(\mathbb{A}\mathbb{B}) \longrightarrow \mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}.$$

Proposition 4.16. *The Ioneda embedding*

$$\iota : (Pro(\mathbb{A}\mathbb{B}))^{op} \longrightarrow \mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$$

commutes with small limits and filtrant colimits.

A morphism

$$f : X \longrightarrow Y$$

in $(Pro(\mathbb{A}\mathbb{B}))^{op}$ is a monomorphism iff $\iota(f)$ is a monomorphism in $\mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$.

Proposition 4.17. (1) *The **contravariant** embedding*

$$\kappa : Pro(\mathbb{A}\mathbb{B}) \longrightarrow \mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$$

converts small colimits in $Pro(\mathbb{A}\mathbb{B})$ to limits in $\mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$. Moreover, morphisms

$$(X_i \longrightarrow X)_{i \in I},$$

where I is a small category, form a colimit in $Pro(\mathbb{A}\mathbb{B})$ iff

$$(\kappa(X) \longrightarrow \kappa(X_i))_{i \in I}$$

form a limit in $\mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$.

(2) *The embedding κ converts cofiltrant limits in $Pro(\mathbb{A}\mathbb{B})$ to filtrant colimits in $\mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$. Moreover, given a small cofiltrant diagram $(X_i)_{i \in I}$, then morphisms*

$$(X \longrightarrow X_i)_{i \in I}$$

form a limit in $Pro(\mathbb{A}\mathbb{B})$ iff

$$(\kappa(X_i) \longrightarrow \kappa(X))_{i \in I}$$

form a colimit in $\mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$.

(3) *A morphism*

$$f : X \longrightarrow Y$$

in $Pro(\mathbb{A}\mathbb{B})$ is an epimorphism iff $\kappa(f)$ is a monomorphism in $\mathbb{A}\mathbb{B}^{\mathbb{A}\mathbb{B}}$.

REFERENCES

- [Bre68] Glen E. Bredon. Cosheaves and homology. *Pacific J. Math.*, 25:1–32, 1968.
- [Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.
- [KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
- [Mar00] Sibe Mardešić. *Strong shape and homology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.
- [MS82] Sibe Mardešić and Jack Segal. *Shape theory*, volume 26 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1982. The inverse system approach.
- [Sch87] Jean-Pierre Schneiders. Cosheaves homology. *Bull. Soc. Math. Belg. Sér. B*, 39(1):1–31, 1987.
- [SGA72] *Théorie des topos et cohomologie étale des schémas. Tome 2*. Lecture Notes in Mathematics, Vol. 270. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.
- [Tam94] Günter Tamme. *Introduction to étale cohomology*. Universitext. Springer-Verlag, Berlin, 1994. Translated from the German by Manfred Kolster.

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